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The symmetries of the Manton superconductivity model

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Abstract

The symmetries and conserved quantities of Manton's modified superconductivity model with non-relativistic Maxwell–Chern–Simons dynamics (also related to the Quantized Hall Effect) are obtained in the "Kaluza–Klein type" framework of Duval et al.

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1. Introduction

Recently [1],[1] Manton proposed a modified version of the Landau–Ginzburg theory of superconductivity. His equations, defined on (2+1)-dimensional non-relativistic space–time parametrized by $\boldsymbol{x}$ and t, read

$$\mathrm{i}\gamma\mathcal{D}_t\Phi = -\tfrac{1}{2}\boldsymbol{\mathcal{D}}^2\Phi - \tfrac{1}{4}(\lambda(1-|\Phi|^2))\Phi, \qquad \text{NLS} \tag{1.1}$$

$$\epsilon_{ij}\partial_j\mathcal{B} = \mathcal{J}_i - J_i^{\mathrm{T}} + 2\kappa\,\epsilon_{ij}\,\mathcal{E}_j, \qquad \text{Ampère–Hall Law} \tag{1.2}$$

$$2\kappa\mathcal{B} = \gamma(1-|\Phi|^2). \qquad \text{Gauss' Law} \tag{1.3}$$

Here $\gamma > 0$, $\lambda > 0$ and $\kappa \in \mathbb{R}$ are constants, $\mathcal{B} = \epsilon^{ij}\partial_i\mathcal{A}^j$ and $\boldsymbol{\mathcal{E}} = \boldsymbol{\nabla}\mathcal{A}_t - \partial_t\boldsymbol{\mathcal{A}}$ are the "statistical" magnetic and the electric fields, respectively, associated with the vector potential $(\mathcal{A}_t, \boldsymbol{\mathcal{A}})$. The covariant derivatives mean $\mathcal{D}_\alpha\Phi = \partial_\alpha\Phi - \mathrm{i}\mathcal{A}_\alpha\Phi$; the current is

$$\mathcal{J}_\alpha = \frac{1}{2i}\left[\Phi^\star\mathcal{D}_\alpha\Phi - \Phi(\mathcal{D}_\alpha\Phi)^\star\right], \tag{1.4}$$

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[1] Further references are found in [12].

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where $\alpha = t, i$. The new ingredient is the transport current $\boldsymbol{J}^{\mathrm{T}}$ (a constant vector).

The matter field Φ satisfies hence a non-linear Schrödinger equation, as in the non-relativistic Chern–Simons theory of Jackiw and Pi [2–5]. The Maxwell term familiar from the static Landau–Ginzburg theory only enters Ampère's law, (1.2), and is missing from Gauss' law, (1.3). In the absence of a magnetic field and of a transport current, Eq. (1.2) reduces to the off-diagonal relation between the current and the electric field

$$\mathcal{J}_i = -2\kappa\, \epsilon_{ij}\, \mathcal{E}_j, \tag{1.2'}$$

which is Hall's law. The Manton model is in fact closely related to the "Landau–Ginzburg" theory of the Quantized Hall Effect (QHE) [6–10], and has indeed been used in this context [11].

The form of the system (1.1)–(1.3) is dictated by the requirement of Galilean covariance [1,12]; it exemplifies a Galilei-invariant electromagnetic theory of the magnetic type [13–15].

To make the magnetic field vanish at infinity, the particle density, $\varrho = |\Phi|^2$, has to tend to 1 rather than to 0 when $r \to \infty$ by Eq. (1.3). Similarly, it follows from Eq. (1.2) that $\boldsymbol{\mathcal{J}} \to \boldsymbol{J}^{\mathrm{T}}$ at infinity.

The system (1.1)–(1.3) admits a surprising six-parameter algebra of symmetries [12,16]. The first three are ordinary space and time translations. The three further symmetries (related to those in a constant external electromagnetic field [17–21] and called "hidden boosts and rotations") are more subtle, see Section 5.

The associated conserved quantities were obtained in [12,16]. The procedure is somewhat tricky in that the naive energy–momentum tensor is not gauge-invariant and the associated integrals do not converge. It has therefore to be "improved". It is natural to inquire about the possibility of obtaining these improved expressions from first principles.

Another surprise is that the momenta satisfy the anomalous commutation relation [22]

$$\{p_1, p_2\} = \gamma \int B\, \mathrm{d}^2x = 2\pi n\gamma, \qquad n \in \mathbb{Z}, \tag{1.5}$$

rather then commute, as ordinary translations do. This relation is very important, since it can be used to explain the quantization of the Hall conductivity in the QHE [9,10].

In this paper, we explain these results using the "non-relativistic Kaluza–Klein-type" framework of Duval et al. [23,24]. The well-known relativistic case is only recalled to motivate our arguments; proofs are only provided in the non-relativistic context, which is our main concern here.

In the approach of [23,24], the $(2+1)$-dimensional dynamics is lifted to a four-dimensional Lorentz manifold (M, g) carrying a covariantly constant light-like vector ξ, referred to as the "Bargmann space". Physics in "ordinary" space is recovered by reduction along ξ. Owing to the singular character of the projection, our systems defined on Bargmann space can only partially be derived from an action principle, forcing us to work mostly with the equations of motion.

In the simplest case (Case A), the Bargmann space is Minkowski space, and we get a variant of the Jackiw–Pi theory [2–5,25]. In Case B, the external fields get included into the metric; the reduction yields the Chern–Simons [17–21,26] theory in external fields. Our clue is that the transport terms behave precisely as external electric and magnetic fields, so that the metric B also describes the Manton model! Then our Theorem 1 states that any ξ-preserving isometry of Bargmann space is a symmetry for the reduced system.

In case A, the ξ-preserving isometries (resp. conformal transformations) of M form the extended Galilei (resp. Schrödinger [27–29]) group [23,24]. For metric B, the conformal transformations form the "hidden Schrödinger algebra" [17–21,26]. Describing the symmetries of the Manton model requires hence selecting the isometries of metric B. These form a *seven*-parameter group, namely those found in [12], augmented with the vertical translations. Thus, the "hidden" symmetries are also "geometric", but with respect to another geometry.

Interestingly, the problem of lifting the symmetries from ordinary space–time to Bargmann space amounts to studying the symmetries in a fixed background field, as discussed by Forgács and Manton [30], and by Jackiw and Manton [31].

The Bargmann framework also allows us to derive a symmetric, conserved energy–momentum tensor. Then the geometric version of Noether's theorem (Theorem 3) associates a conserved quantity to each Killing vector of Bargmann space, yielding, without further "improvement", the same conserved quantities as found before in [12,16]. These facts underline the importance of finding the "good" lift of the space–time transformations.

Our notation are as follows. On ordinary space–time Q: label $\alpha, \beta = t, i$. Vectors $X = (X^\alpha)$; the generators of the Galilei group: upper-case letters, e.g., $\boldsymbol{P} = (P^i)$, $\boldsymbol{G}$, $i = 1, 2$, etc.; generators of the hidden Galilei group: upper-case calligraphic letters, e.g., $\boldsymbol{\mathcal{P}}$, $\boldsymbol{\mathcal{G}}$. Fields: upper-case letters, e.g., A_α, $F_{\alpha\beta}$. Conserved quantities: lower-case letters, e.g., n, h, p_i, $i = 1, 2$, etc. A general Lorentz 4-manifold: $(M, g_{\mu\nu})$. On a Bargmann space $(\hat{M}, \hat{g}_{\mu\nu}, \xi)$ with special metric (2.2) below: "hat" and label $\mu, \nu = t, i, s$. Vectors $\hat{X} = (\hat{X}^\mu)$. For example, lift of an ordinary translation: $\hat{\boldsymbol{P}} = (\hat{P}^\mu)$; lift of a hidden translation $\hat{\boldsymbol{\mathcal{P}}}$. Fields: lower-case letters; e.g., a_μ, $f_{\mu\nu}$, etc. On Minkowski space $\tilde{M}$, $\tilde{g}_{\mu\nu}$: "tilde" and label μ, ν. For example, lift of an ordinary translation: $\tilde{\boldsymbol{P}}$.

2. "A Kaluza–Klein" framework for Maxwell–Chern–Simons theory

2.1. General theory

In relativistic Kaluza–Klein theory [32–34], electromagnetism is described by a Lorentz manifold M; ordinary (relativistic) space–time is the quotient of M by a *space-like* fibration. To get electromagnetism in the plane, we chose M to be $\mathbb{R}^4$ with coordinates x^μ ($\mu = \alpha, 5$, $\alpha = 0, 1, 2$) and the metric

$$(\tilde{g}_{\alpha\beta} + A^{\rm ext}_\alpha A^{\rm ext}_\beta)\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta + \mathrm{d}x^5 (A^{\rm ext}_\alpha\, \mathrm{d}x^\alpha + \mathrm{d}x^5), \tag{2.1}$$

where $\tilde{g}_{\alpha\beta}$ is the (Minkowski) metric on $(2+1)$-dimensional ordinary space–time Q; A_α represents the electromagnetic vector potential. The space-like direction to be factored out is generated by the Killing vector $\xi = \partial_5$.

Gauge theory admits another geometric description, namely using the language of fiber bundles [35–37]. The external electromagnetic field is represented by a connection one-form ϖ on a principal $\mathbb{R}$ (or U(1)) bundle M over space–time Q, whose curvature is the electromagnetic two-form, $\mathrm{d}\varpi = F$. The vector potential A_α is the pull-back of the connection form ϖ by a section of the bundle. This approach makes no reference to any metric, and is therefore valid in both the relativistic and the non-relativistic context.

A Kaluza–Klein type framework for non-relativistic physics was given in [23,24]. Let us consider a 4-manifold M, which is endowed with a Lorentz metric of signature $(-,+,+,+)$ and also carries a covariantly constant null vector, $\xi = (\xi^\mu)$. The quotient of M by the flow of ξ, denoted by Q, is a $(2+1)$-dimensional manifold with a Newton–Cartan structure, i.e., a non-relativistic space–time [23,24]. As found long time ago [38–40], the most general "Bargmann" 4-space has the form

$$g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + 2\,\mathrm{d}t[\mathrm{d}s + (1/\gamma)A_i^{\mathrm{ext}}\,\mathrm{d}x^i] + 2(1/\gamma)A_t^{\mathrm{ext}}\,\mathrm{d}t^2.$$

Here the "transverse metric" g_{ij} as well as the "vector" and "scalar" potentials A_i^{ext} and A_t^{ext}, are functions of "Galilean time" and "position", t and $\boldsymbol{x}$. $\xi = \partial_s$ is a covariantly constant null vector.

In this paper, we only consider Brinkmann metrics with flat transverse space

$$\hat{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + 2\,\mathrm{d}t[\mathrm{d}s + (1/\gamma)A_i^{\mathrm{ext}}\,\mathrm{d}x^i] + 2(1/\gamma)A_t^{\mathrm{ext}}\,\mathrm{d}t^2. \tag{2.2}$$

All such metrics can be viewed as defined on the same manifold (topologically $\mathbb{R}^4$), obtained by distorting the "vertical" components of the Minkowski space metric.

$$\hat{g}_{\mu\nu} = \tilde{g}_{\mu\nu} + \eta_{\mu\nu}, \qquad \tilde{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}\boldsymbol{x}^2 + 2\,\mathrm{d}t\,\mathrm{d}s,$$
$$\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = 2A_\alpha^{\mathrm{ext}}\,\mathrm{d}x^\alpha\,\mathrm{d}t. \tag{2.3}$$

The fields

$$\boldsymbol{E}^{\mathrm{ext}} = -\partial_t \boldsymbol{A}^{\mathrm{ext}} + \boldsymbol{\nabla} A_t^{\mathrm{ext}}, \quad B^{\mathrm{ext}} = \boldsymbol{\nabla}\times\boldsymbol{A}^{\mathrm{ext}}, \tag{2.4}$$

have been interpreted as external electric and magnetic fields, respectively [23,24].

In the relativistic case, the fiber bundle approach can be recovered from that of Kaluza–Klein: the total space, M, is itself a fiber bundle with typical fiber generated by $\xi = \partial_5$,

$$\varpi = \mathrm{i}_\xi g \equiv g(\xi,\,\cdot), \tag{2.5}$$

is a connection form on this bundle. Contracting (2.1) with $\xi = \partial_5$ yields indeed the standard expression $\varpi = A_\alpha^{\mathrm{ext}}\,\mathrm{d}x^\alpha + \mathrm{d}x^5$.

In the non-relativistic case the formula (2.5) breaks down, because the vertical fibration is light-like: $\varpi(\xi) = \xi_\mu\xi^\mu = 0$, contradicting the condition $\varpi(\xi) = 1$ required for a connection form [35]. Put another way, the tangent space to the bundle cannot be decomposed into the direct sum of a horizontal subspace and the vertical subspace since ξ is itself horizontal.

2.2. Non-relativistic Chern–Simons theory

Let us now present our non-relativistic Maxwell–Chern–Simons field theory on M. Let $f = \frac{1}{2} f_{\mu\nu}\, dx^\mu \wedge dx^\nu$ be a closed two-form and $j = (j^\sigma)$ a vector on M. (Locally $f_{\mu\nu} = 2\partial_{[\mu} a_{\nu]}$.) Slightly generalizing the procedure proposed in [25], we

(I) posit the generalized Maxwell–Chern–Simons Field–Current Identities (FCI) on Bargmann space

$$\sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma}\xi^\rho \nabla_\omega f^{\omega\sigma} + 2\kappa f_{\mu\nu} = -\sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma}\xi^\rho j^\sigma, \tag{2.6}$$

where ∇_μ is the covariant derivative w.r.t. the metric $\hat{g}_{\mu\nu}$;

(II) consider the non-linear wave equation for a scalar field ϕ on M,

$$\mathcal{D}_\mu \mathcal{D}^\mu \phi - \frac{R}{6}\phi - 2\frac{\delta U}{\delta \phi^\star} = 0, \tag{2.7}$$

where $\mathcal{D}_\mu$ is the metric and gauge covariant derivative,

$$\mathcal{D}_\mu = \nabla_\mu - \mathrm{i} a_\mu, \tag{2.8}$$

and R is the scalar curvature of the Bargmann space M, and $U = U(|\phi|^2)$ is some scalar potential;

(III) couple Eqs. (2.6) and (2.7) according to

$$j^\mu = \frac{1}{2\mathrm{i}}\left[\phi^\star \mathcal{D}^\mu \phi - \phi(\mathcal{D}^\mu\phi)^\star\right]. \tag{2.9}$$

Requiring ϕ to be equivariant,

$$\xi^\mu \mathcal{D}_\mu \phi = \mathrm{i}\gamma\,\phi, \tag{2.10}$$

this system of equations will project into one on Q. Indeed, contracting Eq. (2.6) with the vertical vector ξ, the antisymmetry implies that $f_{\mu\nu}\xi^\nu = 0$. But $f_{\mu\nu}$ also satisfies, by construction, the homogeneous Maxwell equations $\partial_{[\rho} f_{\mu\nu]} = 0$. Therefore, the Lie derivative of the two-form f by ξ vanishes, $L_\xi f = 0$. It follows that the field strength f is the lift to M of a two-form $F = \frac{1}{2} F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$ on Q. Similarly, using that ξ is covariantly constant, it follows from Eq. (2.6) that $L_\xi j = 0$. The current j^μ projects therefore to one on Q we denote by $J = (J^\alpha)$. Eq. (2.6) descends therefore to Q, providing us with Maxwell–Chern–Simons equations in $(2+1)$ dimensions.

Finally, owing again to equivariance and the form of the metric, the non-linear wave equation (2.7) projects to one on Q.

For simplicity, we only consider the symmetry-breaking fourth-order potential

$$U(|\phi|^2) = \frac{\lambda}{8}\left(1 - |\phi|^2\right)^2. \tag{2.11}$$

2.3. Examples

Let us now consider some examples.

Case A. The simplest choice is Minkowski space, $M = \tilde{M}$,

$$\tilde{g}_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = (\mathrm{d}\boldsymbol{x})^2 + 2\,\mathrm{d}t\,\mathrm{d}s. \qquad \text{(Minkowski space)} \tag{2.12}$$

Then, setting $\Phi = \mathrm{e}^{-\mathrm{i}\gamma s}\phi$, our Eqs. (2.6) and (2.7) reduce to those of Jackiw and Pi [2–5] with an additional magnetic Maxwell term and a symmetry breaking potential,

$$\begin{aligned} &B = -\tfrac{\gamma}{2\kappa}\varrho, \\ &\epsilon_{ij}\partial_j B = J_i + 2\kappa\,\epsilon_{ij}\,E_j, \\ &\mathrm{i}\gamma D_t\Phi = \left[-\tfrac{1}{2}\boldsymbol{D}^2 - \tfrac{\lambda}{4}(1 - |\Phi|^2)\right]\Phi, \end{aligned} \tag{2.13}$$

where $\varrho = \Phi^*\Phi$ and $\boldsymbol{J} = (1)/(2i)[\Phi^*\boldsymbol{D}\Phi - \Phi(\boldsymbol{D}\Phi)^*]$ and $D_\alpha = \partial_\alpha - \mathrm{i}A_\alpha$.

Case B. Let us now consider the special Brinkmann metric $\hat{g}_{\mu\nu}$ in (2.2) on $\mathbb{R}^4$, with

$$\begin{aligned} &A_i^{\text{ext}} = \tfrac{1}{2}\epsilon_{ij}x^j B^{\text{ext}}, \quad && B^{\text{ext}} = \text{const}, \\ &A_t^{\text{ext}} = \boldsymbol{x}\cdot\boldsymbol{E}^{\text{ext}}, && \boldsymbol{E}^{\text{ext}} = \text{const}. \end{aligned} \tag{2.14}$$

Such a metric has vanishing scalar curvature, $R = 0$. Since the only non-vanishing components of the inverse metric are $\hat{g}^{ij}$, $\hat{g}^{is} = -A_i^{\text{ext}}$, $\hat{g}^{ss} = -2A_t^{\text{ext}} - (A^{\text{ext}})^i$ and $\hat{g}^{ts} = 1$, we find that the extra components of the metric simply modify, after reduction, the covariant derivative as

$$\mathcal{D}_\alpha \equiv \partial_\alpha - \mathrm{i}A_\alpha - \mathrm{i}A_\alpha^{\text{ext}} = D_\alpha - \mathrm{i}A_\alpha^{\text{ext}}, \tag{2.15}$$

where $\alpha = t, i$. Our equations become hence

$$\begin{aligned} &B = -\tfrac{\gamma}{2\kappa}\,\varrho, \\ &\epsilon_{ij}\partial_j B = \mathcal{J}_i + 2\kappa\,\epsilon_{ij}\,E_j, \\ &i\gamma\underbrace{(\partial_t - \mathrm{i}A_t - \mathrm{i}A_t^{\text{ext}})}_{\mathcal{D}_t}\Phi = \left[-\tfrac{1}{2}(\underbrace{\boldsymbol{\nabla} - \mathrm{i}\boldsymbol{A} - \mathrm{i}\boldsymbol{A}^{\text{ext}}}_{\boldsymbol{\mathcal{D}}})^2\Phi - \tfrac{\lambda}{4}(1 - |\Phi|^2)\right]\Phi. \end{aligned} \tag{2.16}$$

The fields B and $\boldsymbol{E}$ here only involve the "statistical gauge field" A_α but not the background terms: $B = \epsilon_{ij}\partial_i A_j$, $\boldsymbol{E} = \boldsymbol{\nabla}A_t - \partial_t\boldsymbol{A}$. The external field enters the non-linear Schrödinger equation, though, and also change the current (1.4), $\mathcal{J}_\alpha = J_\alpha - A_\alpha^{\text{ext}}|\Phi|^2$. Consistently with the interpretation of $\boldsymbol{E}^{\text{ext}}$ and B^{ext} in [23,24], these equations describe non-relativistic Chern–Simons vortices in a constant external electric and magnetic field [17–21] (again with an additional magnetic Maxwell term).

This same system admits also another interpretation. For

$$A_t^{\text{ext}} = \frac{1}{2\kappa}\boldsymbol{x}\times\boldsymbol{J}^{\mathrm{T}}, \qquad A_i^{\text{ext}} = -\frac{\gamma}{4\kappa}\,\epsilon_{ij}x^j, \tag{2.17}$$

we get

$$B^{\text{ext}} = \frac{\gamma}{2\kappa}, \qquad E_i^{\text{ext}} = -\frac{\epsilon_{ij}J_j^{\mathrm{T}}}{2\kappa}. \tag{2.18}$$

Setting $\mathcal{A}_\alpha = A_\alpha + A_\alpha^{\text{ext}}$, we have

$$\mathcal{B} = B + \frac{\gamma}{2\kappa}, \quad \mathcal{E}_i = E_i - \epsilon_{ij}J_j^{\mathrm{T}}, \tag{2.19}$$

so that, in terms of the curly quantities, Eq. (2.16) become those of Manton, (1.1)–(1.3)!

3. Variational aspects

It is natural to ask whether the posited field equations come from a variational principle. A system similar to ours was considered by Carroll et al. [41]. Adapting their approach to our case, let us start with a Lorentz 4-manifold M, endowed with a covariantly constant vector ξ^μ, and add tentatively a Chern–Simons type term to the usual matter — Maxwell–Lagrangian, $L = L_1 + L_2$, where

$$\begin{aligned} L_1 &= \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2}(\mathcal{D}_\mu\phi)^* \mathcal{D}^\mu\phi + \frac{R}{12}|\phi|^2 + U(|\phi|^2), \\ L_2 &= \frac{\kappa}{2}\epsilon^{\mu\nu\rho\sigma}\xi_\mu a_\nu f_{\rho\sigma}. \end{aligned} \tag{3.1}$$

The resulting field equations read

$$\partial_\mu f^{\mu\nu} + \kappa\sqrt{-g}\,\epsilon^{\mu\rho\sigma\nu}\xi_\mu f_{\rho\sigma} = \frac{\delta}{\delta a_\nu}(L_1) = -j^\nu, \tag{3.2}$$

supplemented with the matter equation (2.5).

In order to relate this theory to one in one less dimensions, let us assume that ϕ is equivariant, (2.10), and that $f_{\mu\nu}$ is the lift from Q of a two-form $F_{\alpha\beta}$.[2] Hence $f_{\mu\nu}\xi^\mu = 0$.

The field equation (3.2) is similar to those in (2.6), except for the "wrong" position of the $\epsilon_{\mu\nu\rho\sigma}$ tensor. To compare the two theories, let us transfer the $\epsilon_{\mu\nu\rho\sigma}$ to the other side of Eq. (3.2) and contract with ξ^ρ to get, using $f_{\mu\nu}\xi^\mu = 0$,

$$\sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma}\,\xi^\rho\partial_\tau f^{\tau\sigma} + \kappa(\xi_\rho\xi^\rho) f_{\mu\nu} = \sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma}\,\xi^\rho j^\sigma. \tag{3.3}$$

If ξ is *space-like* (or *time-like*), it can be normalized as $\xi_\mu\xi^\mu = \pm 1$. Then Eq. (3.3) is (possibly up to a sign) our Eq. (2.6). In the space-like case, e.g., we get a well-behaved *relativistic* model: the quotient is a Lorentz (or a Euclidean) manifold. Let M be, e.g., Minkowski space with metric $\mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2 + \mathrm{d}w^2$. The vector $\xi = \partial_w$ is space-like and covariantly constant. The quotient is $(2+1)$-dimensional Minkowski space with metric $\mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$ and Eqs. (2.6) and (2.7) project to

$$\begin{aligned} &\partial_\alpha F^{\alpha\gamma} + \kappa\,\epsilon^{\alpha\beta\gamma}F_{\alpha\beta} = J^\gamma, \\ &\mathcal{D}_\alpha\mathcal{D}^\alpha\Phi - 2\tfrac{\delta U}{\delta\Phi^\star} = 0. \end{aligned} \tag{3.4}$$

These are indeed the correct Maxwell–Chern–Simons and Klein–Gordon equations for a relativistic model in $(2+1)$-dimensional flat space [42].

If, however, ξ is *light-like*, $\xi_\mu\xi^\mu = 0$, then the $f_{\mu\nu}$ has a *vanishing* coefficient, and Eq. (3.3) does *not* reproduce those in (1.2) and (1.3).

[2] In the approach presented in I, this follows automatically from the field equation.

In conclusion, (3.1) is a correct Lagrangian in the relativistic case but fails to work in the light-like case, which is precisely our case of interest here. It is worth remarking, however, that the non-linear wave equation, (2.7), is correctly reproduced by variation of the "partial action"

$$S=\int_M\left\{\frac{1}{4}f_{\mu\nu}f^{\mu\nu}+\frac{1}{2}(\mathcal{D}_\mu\phi)^*\,\mathcal{D}^\mu\phi-\frac{1}{2}j^{\mathrm{T}}_\mu j^{\mathrm{T}^\mu}+\frac{R}{12}\,|\phi|^2+U(|\phi|^2)\right\}\sqrt{-g}\,\mathrm{d}^4x. \tag{3.5}$$

In order to make the integral converge, we have added here the (constant) transport term to the Lagrange density

$$-\tfrac{1}{2}j^{\mathrm{T}}_\mu j^{\mathrm{T}^\mu},\quad \text{where}\quad (j^{\mathrm{T}})^t=\gamma(j^{\mathrm{T}})^i=(J^{\mathrm{T}})^i,\quad (j^{\mathrm{T}})^s=0.$$

In fact, $(\mathcal{D}_\mu\phi)^*\mathcal{D}^\mu\phi\ \to\ j^{\mathrm{T}}_\mu j^{\mathrm{T}^\mu}$ as $r\ \to\ \infty$. We also included a Maxwell term for later convenience, see Section 6.

4. Space–time symmetries

4.1. Symmetries in ordinary space

Let us now discuss the space–time symmetries. In the Forgács–Manton–Jackiw approach [30,31], (infinitesimal) symmetries are represented by vector fields $X=(X^\alpha)$ on space–time. In the relativistic context considered by the above authors, these are typically Killing vectors of the space–time metric, which leave the kinetic term invariant and hence act as symmetries for a free system. In the presence of an external electromagnetic field, however, only those vector fields remain symmetries for which the change of the external vector potential A^{ext}_α can be compensated by a suitable gauge transformation,

$$L_XA^{\mathrm{ext}}_\alpha=\partial_\alpha W,\qquad \alpha=i,\ t, \tag{4.1}$$

for some compensating function $W(\boldsymbol{x},t)$. (Owing to the gauge freedom, strict invariance, $L_XA^{\mathrm{ext}}_\alpha=0$, would be too restrictive.) Using the identity

$$(L_XA)_\alpha=\partial_\alpha(A_\beta X^\beta)+X^\beta F_{\beta\alpha},$$

valid for any one-form, this relation is readily seen to be equivalent to

$$F^{\mathrm{ext}}_{\alpha\beta}X^\beta=\partial_\alpha\Upsilon,\quad \Upsilon=A^{\mathrm{ext}}_\alpha X^\alpha-W. \tag{4.2}$$

Note that while A_α and W are gauge-dependent, Υ represents the gauge-independent response of the field to a symmetry transformation [30,31]. The response function Υ also appears in the "spin from isospin contribution" in the associated conservation law, see Section 6.

When several symmetries are present in the theory, they only form a closed algebra when, for any two symmetries X_1 and X_2 and compensating functions $W_1 \equiv W_{X_1}$ and $W_2 \equiv W_{X_2}$, the additional relations

$$L_{X_1} W_2 - L_{X_2} W_1 = W_{[X_1, X_2]}, \tag{4.3}$$

or, equivalently,

$$F^{\text{ext}}_{\alpha\beta} X_1^\alpha X_2^\beta = \Upsilon_{[X_1, X_2]}, \tag{4.3$''$}$$

are also satisfied. Expressed using the transport terms, our conditions (4.1) and (4.3) require

$$\boldsymbol{X} \times \boldsymbol{J}^{\mathrm{T}} = \partial_t \Upsilon, \qquad \epsilon_{ij}(X^t J^{\mathrm{T}^j} - X^j J^{\mathrm{T}^t}) = \partial_i \Upsilon. \tag{4.4}$$

4.2. The bundle picture

The ordinary space approach of Forgács et al. can be translated into fiber bundle language [36,37]. A symmetry of the external electromagnetic field is a vector field $\hat{X} = (\hat{X}^\mu)$ on the bundle which is invariant w.r.t. the action of the structure group on the fibers and which also leaves the connection form invariant,

$$L_{\hat{X}} \varpi = 0. \tag{4.5}$$

In terms of a local section $s : Q \to M$, this condition means precisely (4.1) where $A^{\text{ext}} = s^\star \varpi$. The lift admit the gauge-invariant expression

$$\hat{X} = \bar{X} - \Upsilon^*, \tag{4.6}$$

where $\bar{X}$ is the horizontal lift of X, $\varpi(\bar{X}) = 0$, and Υ^* denotes the fundamental vectorfield [35] associated to Υ. This latter can be recovered from the lift according to

$$\Upsilon = -\varpi(\hat{X}), \tag{4.7}$$

since $\varpi(\Upsilon^\star) = \Upsilon$. The consistency condition (4.3) means that the lifts close into a Lie algebra which acts on $\hat{M}$

$$[\hat{X}_1, \hat{X}_2] = [\widehat{X_1, X_2}]. \tag{4.8}$$

Eq. (4.3') provides in fact a cohomological obstruction for lifting the Lie algebra isomorphically from the base to the bundle [43,44].

4.3. The Kaluza–Klein approach

In the relativistic case, the symmetry conditions (4.1) or (4.2) are readily seen to be equivalent to requiring that the lift $\hat{X}$ be an isometry of the Kaluza–Klein metric,

$$L_{\hat{X}} g = 0. \tag{4.9}$$

In fact, $L_{\hat{X}}\varpi = L_{\hat{X}}(\mathrm{i}_\xi g) = \mathrm{i}_\xi(L_{\hat{X}}g)$. Then the gauge-invariant response of the field to a symmetry transformation is recovered as

$$\Upsilon = g_{\mu\nu}\xi^\mu\hat{X}^\nu = \xi_\nu\hat{X}^\nu. \tag{4.10}$$

Let us now turn to the non-relativistic case. The role of space–time isometries is played here by Galilei transformations.[3] However, since we are only interested in the potential (2.13) which manifestly breaks the conformal transformations, we focus our attention to the isometries. Thus, let $\hat{X} = (\hat{X}^\mu)$ be a Killing vector of the Bargmann metric $\hat{g}_{\mu\nu}$ which also preserves the vertical vector ξ,

$$L_{\hat{X}}\hat{g} = 0, \qquad [\hat{X}, \xi] = 0. \tag{4.11}$$

When ξ is factored out, such a vector projects onto an (infinitesimal) "Galilean isometry" of space–time Q, we denote (with some abuse of notation) by $X = (X^\alpha)$, $(\alpha = t, i)$. In our case this simply means a Galilei transformation of $(2+1)$-dimensional space–time. (The general case is discussed in [23,24].) Conversely, an infinitesimal Galilei transformation X^α on flat space–time, Q lifts, by construction, as the Killing vector $\tilde{X}^\mu$ on Minkowski space $\tilde{M}$. What about a more general Brinkmann metric (2.2)? Let us assume that X^α lifts as a Killing vector $\hat{X}^\mu$ to $(\hat{M}, \hat{g})$. Since $\tilde{X}^\mu$ and $\hat{X}^\mu$ are lifts to $\mathbb{R}^4$ of the same Galilei transformation

$$\hat{X}^\mu = \tilde{X}^\mu + Y^\mu, \tag{4.12}$$

where Y^μ is vertical. In our preferred local frame, we denote the only non-vanishing component of Y^μ by W, $Y^\mu = -W\xi^\mu$. The Lie derivative of the Brinkmann metric (2.2) is hence

$$\begin{aligned}(L_{\hat{X}}\hat{g})_{\mu\nu} = {} & (L_{\tilde{X}}\tilde{g})_{\mu\nu} + \tilde{X}^\rho\partial_\rho\eta_{\mu\nu} + \eta_{\mu\rho}\partial_\nu\tilde{X}^\rho + \eta_{\rho\nu}\partial_\mu\tilde{X}^\rho - \tilde{g}_{\mu s}\partial_\nu W - \tilde{g}_{s\nu}\partial_\mu W \\ & -\eta_{\mu s}\partial_\nu W - \eta_{s\nu}\partial_\mu W.\end{aligned}$$

Here $(L_{\tilde{X}}\tilde{g})_{\mu\nu} = 0$, since $\tilde{X}$ is Killing for Minkowski. The vanishing of $(L_{\hat{X}}\hat{g})_{\mu\nu}$ requires thus

$$\tilde{X}^\rho\partial_\rho\eta_{\mu\nu} - \delta_{\mu o}\partial_\nu W - \delta_{\nu o}\partial_\mu W + \eta_{\mu\rho}\partial_\nu\tilde{X}^\rho + \eta_{\rho\nu}\partial_\mu\tilde{X}^\rho = 0. \tag{4.13}$$

This relation is automatically satisfied with the exception of the components (t, α), for which it requires

$$\tilde{X}^\beta\partial_\beta A^{\mathrm{ext}}_\alpha + A^{\mathrm{ext}}_\beta\partial_\alpha\tilde{X}^\beta = \partial_\alpha W.$$

On the LHS, here we recognize the Lie derivative of A^{ext}_α w.r.t. the "Galilean isometry" $X = (X^\alpha)$, which is hence a symmetry for the external electromagnetic field in the sense of Forgács–Manton–Jackiw [30,31], as anticipated by the notation.

[3] For the purely quartic potential $U = -\frac{1}{8}(\lambda|\phi|^4)$, the Chern–Simons system is symmetric with respect to "non-relativistic conformal transformations" [2–5,25,27–29].

5. Symmetries of the field-theoretical system

Now we prove the following.

Theorem 1. *Any ξ -preserving isometry of Bargmann space is a symmetry of our coupled system of equations (2.6) and (2.7).*

By a symmetry we mean here a transformation which carries a solution into some other solution of the equations of motion.

Our theorem can be shown along the same lines as in [25]. Let us first consider the non-linear wave equation (2.7). It is easy to see that transforming the fields as

$$\delta\phi = L_{\hat{X}}\phi, \qquad \delta a_\mu = L_{\hat{X}} a_\mu,$$

the new fields

$$\phi^* = \phi + \delta\phi, \quad a^*_\mu = a_\mu + \delta a_\mu,$$

are still solutions of (2.7) for all isometries $\hat{X}^\mu$ of $(\hat{M}, \hat{g})$.

Next, the equivariance condition (2.10) plainly requires $\hat{X}^\mu$ to be ξ-preserving. Then the current equation (2.6) behave also correctly.

Finally, let us consider the Chern–Simons equation (2.6). Using

$$\delta f_{\mu\nu} = L_{\hat{X}} f_{\mu\nu} = f_{\mu\rho}\partial_\nu \hat{X}^\rho \hat{X} + f_{\sigma\nu}\partial_\mu \hat{X}^\sigma + \hat{X}^\rho \partial_\rho f_{\mu\nu},$$

Eq. (2.6) becomes, for $f^*_{\mu\nu} = f_{\mu\nu} + \delta f_{\mu\nu}$,

$$2\kappa f^*_{\mu\nu} = -\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}\xi^\rho (j^{\sigma *} + \nabla_\omega f^{\omega\nu *}),$$

i.e.,

$$\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}\xi^\rho \nabla_\omega f^{\omega\nu *} + 2\kappa f^*_{\mu\nu} = -\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}\xi^\rho j^{\sigma *},$$

as required.

To end the general theory, let us point out that Bargmann-conformally related Bargmann manifolds share the same symmetries. This is explained by the geometric version [26] of the “export–import” procedure [17–21], originally due to Niederer [45]. Let us indeed consider two Bargmann spaces $(\hat{M}, \hat{g}, \hat{\xi})$ and $(\tilde{M}, \tilde{g}, \xi)$,[4] and assume that they are Bargmann-conformally related, i.e., there is a differentiable map $\Psi : \hat{M} \to \tilde{M}$ such that

$$\Psi^*\tilde{g} = \Omega^2 \hat{g}, \quad \Psi_* \tilde{\xi} = \hat{\xi}, \tag{5.1}$$

where $\Omega(t, \boldsymbol{x})$ is some positive function. Then, the image by Ψ of any ξ-preserving conformal vectorfield $\hat{X}^\nu$ on $\hat{M}$,

$$(\widetilde{\Psi_* X})^\mu = \frac{\partial \tilde{x}^\mu}{\partial \hat{x}^\nu} X^\nu, \tag{5.2}$$

[4] Note that $\hat{\xi} = \tilde{\xi} = \partial_s = \xi$ generates the vertical translations we denote also by N.

is a $\hat{\xi}$-preserving conformal vectorfield on $\tilde{M}$. (The image of a Killing vector for $\hat{g}$ may not be Killing for $\tilde{g}$, though.) The algebraic structure is preserved by the "exportation"

$$\Psi_*[X_{(1)}, X_{(2)}] = [\Psi_* X_{(1)}, \Psi_* X_{(2)}]. \tag{5.3}$$

5.1. Examples

Let us again consider our examples.

Case A. The $\xi = \partial_s$-preserving conformal vectors of Minkowski space

$$\tilde{X}^\mu = \begin{pmatrix} -\chi t^2 - \rho t - \epsilon \\ \Omega(\boldsymbol{x}) - (\frac{1}{2}\rho + \chi t)\boldsymbol{x} + t\boldsymbol{\beta} + \boldsymbol{\delta} \\ \frac{1}{2}\chi|\boldsymbol{x}|^2 - \boldsymbol{\beta}\cdot\boldsymbol{x} + \eta \end{pmatrix}, \tag{5.4}$$

where $\Omega \in \mathrm{so}(2)$, $\boldsymbol{\beta}, \boldsymbol{\delta} \in \mathbb{R}^2$, $\epsilon, \chi, \rho, \eta \in \mathbb{R}$, interpreted as rotation ($\tilde{R}$), boost ($\tilde{G}$), space translation ($\tilde{P}$), time translation ($\tilde{H}$), expansion ($\tilde{K}$), dilatation ($\tilde{D}$) and vertical translation (N). Calculating the commutation relations shows that this nine-dimensional Lie algebra is indeed the *centrally extended Schrödinger algebra*. The central extension shows up in the commutator of translations with boosts

$$[\underbrace{\text{translation}_i}_{\tilde{P}_i}, \underbrace{\text{boost}_j}_{\tilde{G}_i}] = -\delta_{ij}(\underbrace{\text{vertical translation}}_{N}). \tag{5.5}$$

Projecting the algebra (5.4) into Q (which amounts to keeping just the first two components), we get the eight-dimensional Schrödinger algebra of [27–29]: the central extension is lost under projection.

The ξ-preserving Minkowski space isometries form a seven-dimensional algebra, namely the planar *centrally extended Galilei algebra* (also called the Bargmann algebra), consisting of rotations, boosts, spatial and time translations as well as "vertical" translations (translations along ξ), given by (5.4) with $\chi = \rho = 0$. Projecting the ξ-preserving Killing vectors into Q, we get the planar Galilei algebra with six generators, whose commutation relations differ from those on Bargmann space in that ordinary space boosts and translations commute.

The potential (2.11) breaks the conformal transformations to isometries. Then Theorem 1 implies that the extended Galilei algebra is symmetry for the system (2.13).

Case B. The conformal vectors of metric B form again a nine-dimensional Lie algebra, which is algebraically isomorphic to the Schrödinger algebra. This can either be shown by a lengthy direct calculation, or be derived by the "export–import" procedure [17–21] explained above. Consider the mapping

$$\Psi(t, \boldsymbol{x}, s) = (T, \boldsymbol{X}, S),$$

presented in Eq. (A.1) in Appendix A, constructed of (i) Niederer's transformation [45] which takes the free case to an oscillator, followed by (ii) a rotation which carries the oscillator into a uniform magnetic field, and (iii) followed again by a boost which creates a non-zero electric field. Then Ψ carries the Bargmann space $(\hat{M}, \hat{g}, \xi)$ of Case B into Minkowski space $(\tilde{M}, \tilde{g}, \xi)$ in a ξ-preserving manner.

The image by the inverse mapping Ψ^{-1} of the generators (5.4) $\tilde{X}^\mu$ of the Schrödinger group of Case A is a nine-dimensional algebra defined on the same manifold $\mathbb{R}^4$, made of $\hat{\xi}(=\xi)$-preserving conformal vectors w.r.t. the metric $\hat{g}_{\mu\nu}$ of Case B. By (5.3), these generators satisfy by construction the same commutation relations as their pre-images. We call it therefore "the hidden Schrödinger algebra". These formulae (presented in Appendix A) are rather complicated but nevertheless necessary to establish the crucial relations (5.7) and (5.9) below. These latter provide in turn the "good" lifts (5.8) and (5.10), respectively.

For the quartic potential $-\lambda\Phi^4$, all conformal generators would act as symmetries [17–21,26]. For the symmetry-breaking potential (2.11), only isometries, i.e., solutions of the Killing equation

$$L_{\hat{X}}\hat{g} = 0, \tag{5.6}$$

qualify, though. These are, first of all "hidden translations", $\hat{\boldsymbol{\mathcal{P}}}$, "hidden boosts", $\hat{\boldsymbol{\mathcal{G}}}$, "hidden rotation", $\hat{\mathcal{R}}$ and vertical translation (listed in Eq. (A.2) in Appendix A).

Some of the generators can be replaced by more familiar expressions, though. A certain combination of "hidden translations", "hidden boost" projects in fact to ordinary translations

$$\underbrace{(\text{ordinary translation})_i}_{\hat{P}_i} = \underbrace{(\text{“hidden translation”})_i}_{\hat{\mathcal{P}}_i} + \frac{\gamma}{4\kappa}\epsilon_{ij}\underbrace{(\text{“hidden boost”})_j}_{\hat{\mathcal{G}}_j}, \tag{5.7}$$

showing that we could have traded either the hidden translations or the hidden boosts for ordinary translations and vice versa. The lift to $(\hat{M}, \hat{g}_{\mu\nu})$ of ordinary translations $\boldsymbol{P}$ is therefore

$$\hat{\boldsymbol{P}} = \begin{pmatrix} 0 \\ \boldsymbol{\delta} \\ -\boldsymbol{\delta}\times\dfrac{\boldsymbol{x}}{4\kappa} - \dfrac{\boldsymbol{\delta}\cdot\boldsymbol{J}^{\mathrm{T}}}{\gamma} + \dfrac{t}{2\kappa\gamma}\boldsymbol{\delta}\times\boldsymbol{J}^{\mathrm{T}} \end{pmatrix}. \tag{5.8}$$

Some combination of conformal generators can still be Killing. A look on the explicit expressions (A.4)–(A.6) shows that this happens indeed for

$$\begin{aligned}\underbrace{(\text{hidden time translation})}_{\hat{\mathcal{H}}} &+ (\tfrac{1}{2}B^{\text{ext}})^2 \times \underbrace{(\text{hidden expansion})}_{\hat{\mathcal{K}}} \\ &- (\tfrac{1}{2}B^{\text{ext}}) \times \underbrace{(\text{hidden rotation})}_{\hat{\mathcal{R}}},\end{aligned} \tag{5.9}$$

which project in fact to an *ordinary time translation*, H. This latter lifts therefore to the metric of Case B as

$$\hat{H}=\begin{pmatrix} -\epsilon \\ 0 \\ -\frac{1}{2}\epsilon\left(\dfrac{\boldsymbol{J}^{\mathrm{T}}}{\gamma}\right)^2 \end{pmatrix}. \tag{5.10}$$

Let us note that the formulae referred to above are only valid in the rest frame $\boldsymbol{J}^{\mathrm{T}}=0$; the general formulae can be obtained by a boost.

Finally, the only solutions of the Killing equation (5.6) are combinations of these generators. The conformal vectors of $\hat{g}_{\mu\nu}$ and of $\tilde{g}_{\mu\nu}$ are in fact in bijection by means of the "export–import" map (A.1). But on Minkowski space, the only Bargmann-conformal vectors are those in the Schrödinger algebra. Collecting our results, we state the following.

Theorem 2. *The ξ-preserving isometries of the Bargmann-space of Case B form the seven-dimensional Lie algebra, generated by ordinary space ($\hat{P}$) and time ($\hat{H}$) translations, vertical translations (N), hidden rotations ($\hat{\mathcal{R}}$) and hidden boosts ($\hat{\mathcal{G}}$). Their commutation relations read*

$$\begin{aligned}
&\left[\hat{\mathcal{G}}_i,\hat{\mathcal{G}}_j\right]=0, &&\left[\hat{P}_i,\hat{P}_j\right]=-\tfrac{1}{2\kappa}\epsilon_{ij}N, &&\left[\hat{P}_i,\hat{\mathcal{G}}_j\right]=\delta_{ij}N,\\
&\left[\hat{\mathcal{G}}_i,\hat{\mathcal{R}}\right]=\epsilon_{ij}\hat{\mathcal{G}}_j, &&\left[\hat{P}_i,\hat{\mathcal{R}}\right]=\epsilon_{ij}\hat{P}_j, && \\
&\left[\hat{H},\hat{\mathcal{R}}\right]=0, &&\left[\hat{H},\mathcal{G}_i\right]=\hat{P}_i, &&\left[\hat{H},\hat{P}_i\right]=0.
\end{aligned} \tag{5.11}$$

(The vertical translation, N, commutes with all generators.) These are the commutation relations of the extended Galilei group, with the exception of that, unlike ordinary translations, the lifted translations do not commute. These latters do not form hence a subalgebra on their own but belong rather to a three-parameter subalgebra identified as the Heisenberg algebra, i.e., the central extension of space–time translations with the vertical translation.

Projecting these vector fields into ordinary space–time, we recover the six symmetries found in [12]. By (5.3), the projections satisfy the same commutation relation (5.11), except for that the central extension is lost under the projection, $N\to 0$.

5.2. The lifting problem

Conversely, let us start with the projected vectorfields and lift them to the Bargmann space w.r.t. the metric $\hat{g}_{\mu\nu}$. The lifts have non-trivial fourth components, which come from the condition that the transport terms (*alias* external fields) be symmetric w.r.t. the action on ordinary space–time. Let us illustrate this point on the example of the ordinary space translations. Each of $P_1=\partial_1$ and $P_2=\partial_2$ is a symmetry: the condition (4.2) is verified with

$$\Upsilon_i=-\frac{1}{2\kappa}\epsilon_{ij}(\gamma x^j-tJ_j^{\mathrm{T}})+C_i, \tag{5.12}$$

where C_i is an arbitrary constant. Each of the translations can be lifted therefore *individually* to Bargmann space. No choice of the constants C_i allows to lift the *algebra* of planar translations *isomorphically*, though, since this is forbidden by the cohomological obstruction [43,44] referred to above. Condition (4.4) would require in fact

$$F^{\text{ext}}_{\alpha\beta} P_1^\alpha P_2^\beta = B^{\text{ext}} \equiv \frac{\gamma}{2\kappa} \neq 0.$$

But $\Upsilon_0 = 0$ since P_1 and P_2 commute; a contradiction.

According to (4.1) (or (4.2)), the lift of each symmetry is only unique up to a constant. The ambiguity can be eliminated by requiring that the algebraic structure be (as much as possible) preserved. For example, the Lie bracket

$$[\hat{P}_1, \mathcal{R}] = \hat{P}_2,$$

fixes the constant in $\hat{P}_2$, etc. Note that the constant in the lift of time translations is *not* fixed as long as we only consider the isometries, because $\hat{H}$ is not the Lie bracket of any two isometries, cf. (5.11). For fixing its constant, we must consider the isometries as a subalgebra of the conformal vectors. Then the commutation relation in (A.7) in Appendix A do fix $\hat{H}$ uniquely, as in (5.10), upon use of (5.9). Remarkably, the "good lifts" coincide with those obtained using the "export–import" procedure above. Fixing the lifts plays, as we explain in Section 6, an important role in deriving the conserved quantities.

6. Conserved quantities

The lack of a variational principle forces us to use a mixed approach, presented in [25]. We only consider Case B, since Case A has been discussed in [25]. Applying the method of [46] to the "partial action" S (3.5), yields the symmetric energy–momentum tensor

$$\vartheta_{\mu\nu} = 2\frac{\delta S}{\delta \hat{g}^{\mu\nu}}, \tag{6.1}$$

which also satisfies $\nabla_\mu \vartheta^{\mu\nu} + j_\mu f^{\mu\nu} = 0$. Using our FCI (2.6), we see that the second term vanishes owing to the antisymmetry. The tensor $\vartheta^{\mu\nu}$ is hence itself conserved

$$\nabla_\mu \vartheta^{\mu\nu} = 0. \tag{6.2}$$

Our $\vartheta^{\mu\nu}$ is *not* traceless, though, since the theory is *not* conformally symmetric.

Let $\hat{X}^\mu$ now be a Killing vector of some Brinkmann metric $\hat{g}_{\mu\nu}$, and consider the current

$$k^\mu = \vartheta^\mu_\nu \hat{X}^\nu, \tag{6.3}$$

where k^μ is gauge-invariant by construction. Furthermore

$$\nabla_\mu(\vartheta^\mu_\nu \hat{X}^\nu) = (\nabla_\mu \vartheta^\mu_\nu)\hat{X}^\nu + \tfrac{1}{2}\vartheta^{\mu\nu} L_{\hat{X}} g_{\mu\nu} = 0,$$

since $L_X g_{\mu\nu} = 0$. The current k^μ is therefore conserved, $\nabla_\mu k^\mu = 0$. If $\hat{X}^\mu$ is also ξ-preserving, one can show that the current (k^μ) projects into a three-current

$$(K^\alpha) = (K^t, \boldsymbol{K}), \tag{6.4}$$

on ordinary space–time, Q. The projected current is thus also conserved. Hence we have the following.

Theorem 3. *For each isometry $\hat{X}$, the quantity* [5]

$$\mathcal{Q}_X = \int \vartheta_{\mu\nu}\hat{X}^\mu \xi^\nu \,\mathrm{d}^2\boldsymbol{x}, \tag{6.5}$$

is conserved, provided all currents vanish at infinity.

Remembering that in a local frame the lift $\hat{X}^\mu$ is decomposed as $\hat{X}^\mu = (X^\alpha, -W) = (X^\alpha, -A_\alpha^{\mathrm{ext}} X^\alpha + \Upsilon)$, we get the following.

Corollary. *The conserved quantities admit the gauge-invariant decomposition*

$$\mathcal{Q}_X = \int \underbrace{[\vartheta_{\alpha s} X^\alpha - (A_\alpha^{\mathrm{ext}} X^\alpha)\vartheta_{ss}]}_{\vartheta_{\mu\nu}\tilde{X}^\mu\xi^\nu} \mathrm{d}\boldsymbol{x}^2 + \int \Upsilon \vartheta_{ss}\,\mathrm{d}\boldsymbol{x}^2, \tag{6.6}$$

where Υ is the response of the symmetrical external field to the symmetry $X = (X^\alpha)$ in Eq. (4.2). The second term represents here the contribution of the symmetric external field to the conserved quantity, called the "spin from isospin" phenomenon [30,31,37].

Varying the "partial action" (3.5), a rather tedious calculation similar to that in [25] yields the energy–momentum tensor of the lifted Manton system

$$\begin{aligned}\vartheta_{\mu\nu} = {} & \frac{1}{3}\left((\mathcal{D}_\mu\phi)^*\mathcal{D}_\nu\phi + \mathcal{D}_\mu\phi(\mathcal{D}_\nu\phi)^*\right) - \frac{1}{6}\left(\phi^*\mathcal{D}_\mu\mathcal{D}_\nu\phi + \phi(\mathcal{D}_\mu\mathcal{D}_\nu\phi)^*\right)\\ & +\frac{1}{6}|\phi|^2\left(R_{\mu\nu} - \frac{R}{6}\hat{g}_{\mu\nu}\right) - \frac{1}{6}\hat{g}_{\mu\nu}\left(\mathcal{D}_\sigma\phi(D^\sigma\phi)^*\right) - \frac{1}{4}\hat{g}_{\mu\nu}(f_{\rho\sigma}f^{\rho\sigma}) - f_{\mu\rho}f_\nu^\rho\\ & -\hat{g}_{\mu\nu}\frac{\lambda}{4}\left(-\frac{1}{2} + \frac{1}{3}|\phi|^2 - \frac{1}{6}|\phi|^4\right) - j_\mu^{\mathrm{T}} j_\nu^{\mathrm{T}} + \frac{1}{2}\hat{g}_{\mu\nu}(j_\sigma^{\mathrm{T}} j^{\mathrm{T}^\sigma}).\end{aligned} \tag{6.7}$$

Since each of the currents K^α in (6.4) vanish at infinity, setting $\Lambda = \lambda + (\gamma/\kappa)^2$, Theorem 3 yields the conserved quantities

$$\begin{aligned}
n &= \gamma^2\textstyle\int[1 - |\Phi|^2]\,\mathrm{d}^2\boldsymbol{x} = \gamma\int\left[\frac{\mathcal{B}}{2\kappa}\right]\mathrm{d}^2\boldsymbol{x}, && \text{particle number}\\
p_i &= \gamma\textstyle\int\left[\mathcal{J}_i - J_i^{\mathrm{T}}|\phi|^2 + \left\{\epsilon_{ij}\left(x^j - t\frac{J_j^{\mathrm{T}}}{\gamma}\right)\right\}\mathcal{B}\right]\mathrm{d}^2\boldsymbol{x}, && \text{momentum}\\
h &= \textstyle\int\Big[\frac{1}{2}|\boldsymbol{D}\phi|^2 - \frac{1}{2}|\boldsymbol{J}^{\mathrm{T}}|^2|\phi|^2 + \frac{\Lambda}{8}(1 - |\phi|^2)^2\\
&\quad +\{-(\boldsymbol{x}\times\boldsymbol{J}^{\mathrm{T}})\}\mathcal{B}\Big]\,\mathrm{d}^2x, && \text{energy}\\
m &= \gamma\textstyle\int\Big[\boldsymbol{x}\times(\boldsymbol{\mathcal{J}} - \boldsymbol{J}^{\mathrm{T}}|\phi|^2) - \frac{t}{\gamma}\boldsymbol{J}^{\mathrm{T}}\times\boldsymbol{\mathcal{J}}\\
&\quad + \left\{-\frac{1}{2}r^2 + \frac{t}{\gamma}(\boldsymbol{x}\cdot\boldsymbol{J}^{\mathrm{T}}) - \frac{1}{2}\left(\frac{t}{\gamma}\right)^2|\boldsymbol{J}^{\mathrm{T}}|^2\right\}\mathcal{B}\Big]\,\mathrm{d}^2\boldsymbol{x}. && \text{hidden angular momentum}
\end{aligned} \tag{6.8}$$

[5] The notation $\mathcal{Q}_X$ is, strictly speaking, an abuse, since the conserved quantity actually depends on the lift $\hat{X}^\mu$, and not only on the space–time vector X^α.

(The conserved quantities associated with "hidden boosts" are not illuminating and are therefore not reproduced here.)

These quantities, obtained here from first principles and without any further "improvement" are identical to those found before [12,16]. Note that in the frame $\boldsymbol{J}^{\mathrm{T}} = 0$ our $\mathcal{M}$ becomes the ordinary angular momentum in [12,16]. These expressions nicely illustrate the "spin from isospin" phenomenon [30,31]: the expressions in the curly brackets are the Υs in the symmetry definition (4.2).

The Poisson brackets of the conserved quantities (6.8) were calculated in [12]. They verify the same commutation relations as the ξ-preserving isometries of metric B, listed in Eq. (5.11). The algebraic structure of the conserved quantities reflects hence that of Bargmann space vectors: in Souriau's terminology [43], the "moment map" is equivariant for the centrally extended algebra rather than for the projected algebra. For the momenta in particular, the anomalous commutation relation (1.5) is recovered. (This latter relation can also be understood by observing that conserved quantities associated to "hidden translations" and "hidden boosts" satisfy (5.7).)

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Appendix A

The Bargmann-conformal transformation $\Psi(t, \boldsymbol{x}, s) \to (T, \boldsymbol{X}, S)$ which takes Metric B into that of Minkowski space reads explicitly

$$
\begin{aligned}
T &= \frac{2}{B^{\mathrm{ext}}} \tan \frac{2}{\gamma B^{\mathrm{ext}}}\, t, \\
X_k &= x_k - \epsilon_{kl} \frac{E_l^{\mathrm{ext}}}{B^{\mathrm{ext}}}\, t - \epsilon_{kl} \left(x_l - \epsilon_{lm} \frac{E_m^{\mathrm{ext}}}{B^{\mathrm{ext}}}\, t \right) \tan \frac{2}{\gamma B^{\mathrm{ext}}}\, t, \\
S &= s + \epsilon_{lm} x_l \frac{E_m^{\mathrm{ext}}}{B^{\mathrm{ext}}} - \frac{1}{\kappa} \left(\frac{\boldsymbol{E}^{\mathrm{ext}}}{B^{\mathrm{ext}}} \right)^2 t + \frac{1}{2\gamma}\, t\, \boldsymbol{x} \cdot \boldsymbol{E}^{\mathrm{ext}} \\
&\quad - \frac{1}{2\gamma} B^{\mathrm{ext}} \left(x_l - \epsilon_{lm} \frac{E_m^{\mathrm{ext}}}{B^{\mathrm{ext}}}\, t \right)^2 \tan \frac{2}{\gamma B^{\mathrm{ext}}} t.
\end{aligned}
\tag{A.1}
$$

The "hidden Schrödinger algebra" is obtained by "importing" the Schrödinger algebra (5.4) by (A.1). The isometries act on Bargmann space as

$$\hat{\boldsymbol{\mathcal{P}}} = \cos\frac{1}{4\kappa}t\begin{pmatrix} 0 \\ \cos\dfrac{1}{4\kappa}t\Gamma_1 + \sin\dfrac{1}{4\kappa}t\Gamma_2 \\ -\sin\dfrac{1}{4\kappa}t\Gamma_1 + \cos\dfrac{1}{4\kappa}t\Gamma_2 \\ f \end{pmatrix} \quad \text{“hidden translations”},$$

$$\hat{\boldsymbol{\mathcal{G}}} = \frac{4\kappa}{\gamma}\sin\tfrac{1}{4\kappa}t\begin{pmatrix} 0 \\ \cos\dfrac{1}{4\kappa}t\beta_1 + \sin\dfrac{1}{4\kappa}t\beta_2 \\ -\sin\dfrac{1}{4\kappa}t\beta_1 + \cos\dfrac{1}{4\kappa}t\beta_2 \\ g \end{pmatrix} \quad \text{“hidden boosts”}, \tag{A.2}$$

$$\hat{\mathcal{R}} = \begin{pmatrix} 0 \\ \Omega\left(-x_2 + J_2^{\mathrm{T}}\dfrac{t}{\gamma}\right) \\ \Omega\left(x_1 - J_1^{\mathrm{T}}\dfrac{t}{\gamma}\right) \\ h \end{pmatrix} \quad \text{“hidden rotations”},$$

$$N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta \end{pmatrix} \quad \text{“vertical translations”},$$

where $\boldsymbol{\Gamma}, \boldsymbol{\beta} \in \mathbb{R}^2$, $\Omega, \eta \in \mathbb{R}$. In these formulae, f, g and h are shorthands for the complicated expressions

$$\begin{aligned}
f &= \frac{1}{4\kappa\cos\dfrac{1}{4\kappa}t}\left[-\sin^2\left(\frac{t}{4\kappa}\right)\boldsymbol{\gamma}\times\boldsymbol{x} + \sin\left(\frac{t}{4\kappa}\right)\cos\left(\frac{t}{4\kappa}\right)\boldsymbol{x}\cdot\boldsymbol{\Gamma} + \frac{t}{\gamma}\boldsymbol{\Gamma}\times\boldsymbol{J}^{\mathrm{T}}\right. \\
&\left. -\frac{4\kappa}{\gamma}\cos^2\left(\frac{t}{4\kappa}\right)\boldsymbol{\gamma}\cdot\boldsymbol{J}^{\mathrm{T}} + \frac{4\kappa}{\gamma}\sin\left(\frac{t}{4\kappa}\right)\cos\left(\frac{t}{4\kappa}\right)\boldsymbol{\Gamma}\times\boldsymbol{J}^{\mathrm{T}}\right], \\
g &= \frac{1}{4\kappa\sin\dfrac{1}{4\kappa}t}\left[-\cos^2\left(\frac{t}{4\kappa}\right)\boldsymbol{x}\cdot\boldsymbol{\beta} + \sin\left(\frac{t}{4\kappa}\right)\cos\left(\frac{t}{4\kappa}\right)\boldsymbol{\beta}\times\boldsymbol{x} + \frac{t}{\gamma}\boldsymbol{\beta}\cdot\boldsymbol{J}^{\mathrm{T}}\right. \\
&\left. +\frac{4\kappa}{\gamma}\sin^2\left(\frac{t}{4\kappa}\right)\boldsymbol{\beta}\times\boldsymbol{J}^{\mathrm{T}} - \frac{4\kappa}{\gamma}\sin\left(\frac{t}{4\kappa}\right)\cos\left(\frac{t}{4\kappa}\right)\boldsymbol{\beta}\cdot\boldsymbol{J}^{\mathrm{T}}\right], \\
h &= \Omega\left[-\frac{t}{4\kappa\gamma}(\boldsymbol{x}\cdot\boldsymbol{J}^{\mathrm{T}}) + \frac{1}{4\kappa}\left(\frac{t}{\gamma}\right)^2(\boldsymbol{J}^{\mathrm{T}})^2 - \frac{\boldsymbol{x}\times\boldsymbol{J}^{\mathrm{T}}}{\gamma}\right].
\end{aligned} \tag{A.3}$$

"Hidden dilatations" and "hidden expansions" and (somewhat surprisingly) "hidden time translations" are conformal but not Killing. Setting $\tau = t/4\kappa$, they read [6]

$$\hat{\mathcal{H}} = \text{hidden time translation} = \begin{pmatrix} -\gamma\cos^2\tau \\ \dfrac{\gamma}{4\kappa}\cos\tau\,(x_1\sin\tau - x_2\cos\tau) \\ \dfrac{\gamma}{4\kappa}\cos\tau\,(x_1\cos\tau + x_2\sin\tau) \\ -\dfrac{r^2\gamma}{32\kappa^2}\cos 2\tau \end{pmatrix}. \tag{A.4}$$

$$\hat{\mathcal{K}} = \text{hidden expansion} = \begin{pmatrix} -\left(\dfrac{16\kappa^2}{\gamma}\right)\sin^2\tau \\ -\dfrac{4\kappa}{\gamma}\sin\tau\,(x_1\cos\tau + x_2\sin\tau) \\ \dfrac{4\kappa}{\gamma}\sin\tau\,(x_1\sin\tau - x_2\cos\tau) \\ \dfrac{r^2}{2\gamma}\cos 2\tau \end{pmatrix}. \tag{A.5}$$

$$\hat{\mathcal{D}} = \text{hidden dilatation} = -\frac{1}{2}\begin{pmatrix} 4\kappa\sin 2\tau \\ x_1\cos 2\tau - x_2\sin 2\tau \\ x_1\sin 2\tau + x_2\cos 2\tau \\ \dfrac{r^2}{4\kappa}\sin 2\tau \end{pmatrix}. \tag{A.6}$$

The commutation relations of the "hidden" quantities are those of the Schrödinger algebra:

$$\begin{aligned}
&[\mathcal{G}_i, \mathcal{G}_j] = 0, \quad [\mathcal{P}_i, \mathcal{P}_j] = 0, \quad [\mathcal{P}_i, \mathcal{G}_j] = \frac{\delta_{ij}}{\gamma}N,\\
&[\mathcal{G}_i, \mathcal{R}] = \epsilon_{ij}\mathcal{G}_j, \quad [\mathcal{P}_i, \mathcal{R}] = \epsilon_{ij}\mathcal{P}_j, \quad [\mathcal{H}, \mathcal{G}_i] = \mathcal{P}_i,\\
&[\mathcal{H}, \mathcal{P}_i] = 0, \quad [\mathcal{H}, \mathcal{R}] = 0, \quad [\mathcal{H}, \mathcal{D}] = 2\mathcal{H},\\
&[\mathcal{H}, \mathcal{K}] = \mathcal{D}, \quad [\mathcal{D}, \mathcal{K}] = 2\mathcal{K}, \quad [\mathcal{R}, \mathcal{D}] = 0,\\
&[\mathcal{R}, \mathcal{K}] = 0, \quad [\mathcal{D}, \mathcal{G}_i] = \mathcal{G}_i, \quad [\mathcal{D}, \mathcal{P}_i] = -\mathcal{P}_i,\\
&[\mathcal{K}, \mathcal{G}_i] = 0, \quad [\mathcal{K}, \mathcal{P}_i] = \mathcal{G}_i.
\end{aligned} \tag{A.7}$$

[6] For simplicity, we only present the formulae valid in the rest frame $\boldsymbol{J}^{\mathrm{T}} = 0$. The general expressions (which would take several pages to write) can be found by boosting those in (A.4).

References

[1] N.S. Manton, First order vortex dynamics, Ann. Phys. (NY) 25 (1997) 114.
[2] R. Jackiw, S.-Y. Pi, Soliton solutions to the gauged nonlinear Schrödinger equation on the plane, Phys. Rev. Lett. 64 (1990) 2969.
[3] R. Jackiw, S.-Y. Pi, Classical and quantal non-relativistic Chern–Simons theory, Phys. Rev. D 42 (1990) 3500.
[4] R. Jackiw, S.-Y. Pi, Self-dual Chern–Simons solitons, Prog. Theor. Phys. Suppl. 107 (1992) 1.
[5] G. Dunne, Self-dual Chern–Simons theories, Lecture Notes in Physics, New Series: Monograph 36, Springer, Berlin, 1995.
[6] S.M. Girvin, Towards a Landau–Ginsburg theory of the FQHE, in: R.E. Prange, S.M. Girvin (Eds.), The Quantum Hall Effect, Springer, New York, 1986 (Chapter 10).
[7] S.M. Girvin, A.-H. MacDonald, Off-diagonal long-range order, oblique confinement, and the fractional quantum Hall effect, Phys. Rev. Lett. 58 (1987) 303.
[8] S.C. Zhang, T.H. Hanson, S. Kivelson, Effective-field theory model for the fractional quantum Hall effect, Phys. Rev. Lett. 62 (1989) 307.
[9] G. Morandi, Topological problems in condensed matter physics, Quantum Hall Effect, Bibliopolis, Napels, 1988.
[10] M. Stone (Ed.), Quantum Hall Effect, World Scientific, Singapore, 1992.
[11] M. Hassaïne, P.A. Horváthy, J.-C. Yera, Vortices in the Landau–Ginzburg model of the quantized Hall effect, J. Phys. A 31 (1998) 9073–9079.
[12] M. Hassaïne, P.A. Horváthy, J.-C. Yera, Non-relativistic Maxwell–Chern–Simons vortices, Ann. Phys. (NY) 263 (1998) 276–294.
[13] M. Le Bellac, J.-M. Lévy-Leblond, Galilean electromagnetism, Il Nuovo Cimento B 14 (1973) 217.
[14] C.R. Hagen, A new gauge theory without an elementary photon, Ann. Phys. (NY) 157 (1984) 342.
[15] C.R. Hagen, Galilean-invariant gauge theory, Phys. Rev. D 31 (1985) 848.
[16] N.S. Manton, S.M. Nasir, Conservation laws in a first-order dynamical system of vortices, Nonlinearity 12 (1999) 851–865.
[17] Z.F. Ezawa, M. Hotta, A. Iwazaki, Breathing vortex solitons in nonrelativistic Chern–Simons gauge theory, Phys. Rev. Lett. 67 (1991) 411.
[18] Z.F. Ezawa, M. Hotta, A. Iwazaki, Nonrelativistic Chern–Simons vortex solitons in external magnetic field, Phys. Rev. D 44 (1991) 452.
[19] R. Jackiw, S.-Y. Pi, Semiclassical Landau levels of anyons, Phys. Rev. Lett. 67 (1991) 415.
[20] R. Jackiw, S.-Y. Pi, Time-dependent Chern–Simons solitons and their quantization, Phys. Rev. D 44 (1991) 2524.
[21] M. Hotta, Imported symmetry and two breathing modes in Chern–Simons theory with external magnetic field, Prog. Theor. Phys. 86 (1991) 1289.
[22] I. Barashenkov, A. Harin, Non-relativistic Chern–Simons theory for the repulsive Bose gas, Phys. Rev. Lett. 72 (1994) 1575.
[23] C. Duval, G. Burdet, H.-P. Künzle, M. Perrin, Bargmann structures and Newton–Cartan theory, Phys. Rev. D 31 (1985) 1841.
[24] C. Duval, G. Gibbons, P. Horváthy, Celestial mechanics, conformal structures, and gravitational waves, Phys. Rev. D 43 (1991) 3907.
[25] C. Duval, P.A. Horváthy, L. Palla, Conformal symmetry of the coupled Chern–Simons and gauged nonlinear Schrödinger equations, Phys. Lett. B 325 (1994) 39.
[26] C. Duval, P.A. Horváthy, L. Palla, Conformal properties of Chern–Simons vortices in external fields, Phys. Rev. D 50 (1995) 6658.
[27] R. Jackiw, Scaling symmetries, Phys. Today 25 (1972) 23.
[28] U. Niederer, The maximal kinematical invariance group of the free Schrödinger equation, Helvetica Physica Acta 45 (1972) 802.
[29] C.R. Hagen, Scale and conformal transformations in Galilean-covariant field theory, Phys. Rev. D 5 (1972) 377.
[30] P. Forgács, N. Manton, Space–time symmetries in gauge theories, Commun. Math. Phys. 72 (1980) 15.
[31] R. Jackiw, N. Manton, Symmetries and conservation laws in gauge theories, Ann. Phys. (NY) 127 (1980) 257.
[32] T. Kałuza, Sitzungsber. Preuss. Acad. Wiss. Phys. Math. K 1 (1916) 996.

[33] O. Klein, Z. Phys. 37 (1926) 895.
[34] R. Coquereaux, A. Jadczyk, Riemannian geometry, fiber bundles, Kaluza–Klein theories and all that, World Scientific Lecture Notes, vol. 16, World Scientific, Singapore, 1988.
[35] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. I, Interscience, New York, 1963.
[36] J. Harnad, S. Shnider, L. Vinet, Group action on principal bundles and invariance conditions for gauge fields, J. Math. Phys. 21 (1980) 2719.
[37] C. Duval, P.A. Horváthy, Particles with internal structure: the geometry of classical motions and conservation laws, Ann. Phys. (NY) 142 (1982) 10–33.
[38] H.W. Brinkmann, Einstein spaces which are mapped conformally on each other, Math. Ann. 94 (1925) 119.
[39] C.R. Nappi, E. Witten, Wess–Zumino model based on a nonsemisimple group, Phys. Rev. Lett. 71 (1993) 3751.
[40] C. Duval, Z. Horváthy, P.A. Horváthy, Strings in plane-fronted gravitational waves, Mod. Phys. Lett. A 8 (1993) 3749–3756 .
[41] S.M. Carroll, G.B. Field, R. Jackiw, Limits on a Lorentz and parity violating modification of electrodynamics, Phys. Rev. D 41 (1990) 1231.
[42] S.K. Paul, A. Khare, Charged vortices in the abelian Higgs model with Chern–Simons term, Phys. Lett. B 174 (1986) 420.
[43] J.M. Souriau, Structure des Systèmes Dynamiques, Dunod, Paris, 1970.
[44] D.J. Simms, Projective representations, symplectic manifolds and extensions of Lie algebras, Marseille Lecture Notes CPT-69/P.300, 1969.
[45] U. Niederer, The maximal kinematical invariance group of the harmonic oscillator, Helvetica Physica Acta 46 (1973) 191.
[46] J.-M. Souriau, Modèle de particule à spin dans le champ électromagnétique et gravitationnel, Ann. Inst. H. Poincaré 20A (1974) 315.